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**Leibniz algebras, Lie racks, and digroups.** **(English summary)**
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In this paper the author, looking at an appropriate generalization of Lie's third theorem for Leibniz algebras, examines objects that play a role analogous to that played by Lie groups for Lie algebras (for which Leibniz algebras are the corresponding tangent algebra structure). This problem, known as the "coquecigrue" problem, was stated by J.-L. Loday [Enseign. Math. (2) **39** (1993), no. 3-4, 269–293; MR1252069].

Two interesting generalizations for Leibniz algebras that split over an ideal containing its ideal generated by squares are obtained. Lie rack is a notion given in the first approach; the notion of Lie digroup is given in the second one.

Let $\mathfrak{g}$ be a Leibniz algebra and $\mathcal{S}\subset\mathfrak{g}$ the ideal generated by all squares. We say that the Leibniz algebra $\mathfrak{g}$ splits over an ideal $\mathcal{E}\subset\mathfrak{g}$ such that $\mathcal{S}\subset\mathcal{E}\subset\ker(\mathrm{ad})$ if there exists a Lie subalgebra $\mathfrak{h}\subset\mathfrak{g}$ such that $\mathfrak{g}=\mathcal{E}\oplus\mathfrak{h}$. A pointed rack is a system $(Q,\circ,1)$ where $Q$ is a set with a binary operation $\circ\colon Q\times Q\to Q$ and a distinguished element $1\in Q$ such that the following three axioms are satisfied: (1) $x\circ(y\circ z)=(x\circ y)\circ(x\circ z)$ for all $x,y,z\in Q$, (2) for each $a,b\in Q$, there exists a unique $x\in Q$ such that $a\circ x=b$, (3) $1\circ x=x$ and $x\circ 1=1$ for all $x\in Q$. A Lie rack is a smooth manifold $Q$ with the structure of a pointed rack such that the rack operation $\circ\colon Q\times Q\to Q$ is a smooth mapping. A digroup $(G,\vdash,\dashv)$ is a dimonoid in the sense of Loday [in *Dialgebras and related operads*, 7–66, Lecture Notes in Math., 1763, Springer, Berlin, 2001; MR1860994] with a bar-unit 1 ($x\dashv 1=x=1\vdash x$ for all $x\in G$) and for each $x$ from $G$ there exists $x^{-1}\in G$ such that $x\vdash x^{-1}=1=x^{-1}\dashv x$. A Lie digroup $(G,\vdash,\dashv)$ is a smooth manifold $G$ with the structure of a digroup such that the digroup operations $\vdash,\dashv\colon G\times G\to G$ and the inversion $(\cdot)^{-1}\colon G\to G$ are smooth mappings.

The author shows that for an arbitrary Lie rack there exists a structure of a Leibniz algebra on the tangent space at the distinguished unit element. The main idea is to differentiate the conjugation operation to obtain the adjoint representation of the group, and then differentiate again to obtain a mapping which is used to define the Lie bracket. Conversely, for a split Leibniz algebra $\mathfrak{g}$ there exists a Lie rack $(Q,\circ,1)$ with the tangent space $T_1Q$ isomorphic to $\mathfrak{g}$. Lie racks are not a "sufficient answer" to the coquecigrue problem. The author demonstrates that any Leibniz algebra can be obtained as the tangent Leibniz algebra of some rack, but in the Lie algebra case, the Lie rack obtained is not a conjugation rack of the Lie group.

Digroups give a more appropriate approach to finding a "correct Lie's third theorem" for Leibniz algebras. Using the foundations of semigroup theory, it is shown that every digroup is a product of a group and a trivial digroup and that for every split Leibniz algebra $\mathfrak{g}$ there exists a linear Lie digroup $G$ with tangent Leibniz algebra $T_1G$ isomorphic to $\mathfrak{g}$, where 1 is a distinguished unit element in $G$.

Not every Leibniz algebra splits [see Y. Kosmann-Schwarzbach, Lett. Math. Phys. **69** (2004), 61–87; MR2104437], so the general coquecigrue problem for Leibniz algebras is open.

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*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*